Investigation of the sum of series from the general Term *

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§103 Let the general term corresponding to the index x of a certain series be $= y$, so that y is a function of x. Further, let Sy be the sum or the summatory term of the series expressing the aggregate of all terms from the first or another fixed term up to y. Indeed, in the following we will compute the sum of the series starting from the first term, whence, if $x = 1$, y will give the first term and Sy will exhibit this first term y; but if one puts $x = 0$, the summatory term Sy has to go over into zero, because there are no terms to be summed. Therefore, the summatory term Sy will be a function of x vanishing for $x = 0$.

§104 If the general term y consists of several parts so that $y = p + q + r +$ etc., then one can consider the series as conflated of several other series, whose general terms are p, q, r etc. Hence, if the sums of these series are known, one will also be able to assign the sum of the propounded series; for, it will be the aggregate of all single series. Therefore, if $y = p + q + r +$ etc., it will be $Sy = Sp + Sq + Sq +$ etc. Therefore, because above we exhibited the sums of series, whose general terms are arbitrary positive powers of x with positive integer coefficients, hence one will be able to find the summatory term of any series, whose general term is $ax^{\alpha} + bx^{\beta} + cx^{\gamma} +$ etc., while α, β, γ etc. are positive integer numbers, or whose general term is a polynomial function of x.

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§105 In this series whose general term or the term corresponding to the exponent x is $= y$ let the term preceding this one or the term corresponding to the index $x-1$ be $= v$; since v results from y, if one writes $x-1$ instead of x, it will be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

Therefore, if y was the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4\cdots & \cdots x-1 & x \\ a+ & b+ & c+ & d+ & \cdots + v+ & y \end{array}$$

and the term corresponding to the index 0 of this series was $= A$, v, since it is a function of x, will be the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5\cdots\cdots\cdots & x \\ A+ & a+ & b+ & c+ & d\ +\ \cdots & +\, v, \end{array}$$

whence, if Sv denotes the sum of this series, it will be $Sv = Sy - y + A$. Therefore, having put $x = 0$, since $Sy = 0$ and $y = A$, also Sv will vanish.

§106 Therefore, since

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.},$$

by means of the results demonstrated before, it will be

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

and, because of $Sv = Sy - y + A$, it will be

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{d^4y}{24dx^4} + \text{etc.}$$

and hence one will have

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

Therefore, if one knows the summatory terms of the series, whose general terms are $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc., from them one will obtain the summatory term of the series, whose general term is $\frac{dy}{dx}$. The quantity A has to be of such a nature that for $x = 0$ the summatory $S\frac{dy}{dx}$ term vanishes, and using this condition, it is determined more easily than if we would say that it is the term corresponding to the index 0 in the series whose general term is $= y$.

§107 This ansatz is usually made to investigate the sums of the powers of natural numbers. For, let $y = x^{n+1}$; since

$$\frac{dy}{dx} = (n+1)x^n, \quad \frac{ddy}{2dx^2} = \frac{(n+1)n}{1\cdot 2}x^{n-1}, \quad \frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}x^{n-2},$$

$$\frac{d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}x^{n-3} \quad \text{etc.},$$

having substituted these values, it will be

$$(n+1)Sx^n = x^{n+1} - A + \frac{(n+1)n}{1\cdot 2}Sx^{n-1} - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}Sx^{n-2} + \text{etc.};$$

and if one divides by $n+1$ on both sides, it will be

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{n}{2}Sx^{n-1} - \frac{n(n-1)}{2\cdot 3}Sx^{n-2} + \frac{n(n-1)(n-2)}{2\cdot 3\cdot 4}Sx^{n-3} - \text{etc.} - \text{Const.},$$

which constant has to be taken in such a way that for $x = 0$ the summatory term vanishes. Therefore, using this formula, from the already known sums of lower powers, whose general terms are x^{n-1}, x^{n-2} etc., one will be able to find the sum of the higher powers expressed by the general term x^n.

§108 If in this expression n denotes a positive integer, the number of terms will be finite. And hence the sum of infinitely many powers will be found explicitly; for, if $n = 0$, it will be

$$Sx^0 = x.$$

And having known this one, it will be possible to proceed to the sums of higher powers; for, having put $n = 1$, it will be

$$Sx^1 = \frac{1}{2}x^2 + \frac{1}{2}Sx^0 = \frac{1}{2}x^2 + \frac{1}{2}x;$$

if one further sets $n = 2$, this equation will result

$$Sx^2 = \frac{1}{3}x^3 + Sx - \frac{1}{3}Sx^0 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,$$

moreover,

$$Sx^3 = \frac{1}{4}x^4 + \frac{3}{2}Sx^2 - Sx + \frac{1}{4}Sx^0 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2,$$

$$Sx^4 = \frac{1}{5}x^5 + \frac{4}{2}Sx^3 - \frac{4}{2}Sx^2 + Sx - \frac{1}{5}Sx^0$$

or

$$Sx^4 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x.$$

And so forth, the successive sums of all higher powers are derived from the lower ones; but the same is achieved a lot easier as follows.

§109 Since we found above that

$$S\frac{dy}{dx} = y + \frac{1}{2}S\frac{ddy}{dx^2} - \frac{1}{6}S\frac{d^3y}{dx^3} + \frac{1}{24}S\frac{d^4y}{dx^4} - \frac{1}{120}S\frac{d^5y}{dx^5} + \text{etc.}$$

if we put $\frac{dy}{dx} = z$, it will be $\frac{ddy}{dx^2} = \frac{dz}{dx}$, $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$ etc. But then, because of $dy = zdx$, y will be the quantity whose differential is $= zdx$ which we denote by $y = \int zdx$. Although this way to find y from given z depends on integral calculus, we will nevertheless be able to use this formula $\int zdx$ here, if we substitute only functions of x of such a kind for z that this function whose differential is $= zdx$ can be exhibited from the preceding ones. Therefore, having substituted these values, it will be

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.},$$

adding such a constant that for $x = 0$ the sum Sz also vanishes.

§110 But by substituting the letter z for y in the above expression or, which is the same, by differentiating this equation it will be

$$S\frac{dz}{dx} = z + \frac{1}{2}S\frac{ddz}{dx^2} - \frac{1}{6}S\frac{d^3z}{dx^3} + \frac{1}{24}S\frac{d^4z}{dx^4} - \text{etc.};$$

but if one writes $\frac{dz}{dx}$ instead of y, it will be

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2}S\frac{d^3z}{dx^3} - \frac{1}{6}S\frac{d^4z}{dx^4} + \frac{1}{24}S\frac{d^5z}{dx^5} - \text{etc.}$$

But if in like manner one successively substitutes the values $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. for y, one will find

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \frac{1}{2}S\frac{d^4z}{dx^4} - \frac{1}{6}S\frac{d^5z}{dx^5} + \frac{1}{24}S\frac{d^6z}{dx^6} - \text{etc.}$$

$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^3} + \frac{1}{2}S\frac{d^5z}{dx^5} - \frac{1}{6}S\frac{d^6z}{dx^6} + \frac{1}{24}S\frac{d^7z}{dx^7} - \text{etc.}$$

and so forth to infinity.

§111 If now these values are successively substituted for $S\frac{dz}{dx}$, $S\frac{ddz}{dx^2}$, $S\frac{d^3z}{dx^3}$ etc. in the expression

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.},$$

one will find an expression for Sz consisting of these terms $\int zdx$, z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., whose coefficients are investigated more easily the following way. Put

$$Sz = \int zdx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.}$$

and for these terms substitute their values they obtain from the preceding series, from which

$$
\begin{aligned}
\int z dx &= Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \text{etc.}\\
\alpha z &= \qquad + \alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\beta dz}{dx} &= \qquad\qquad + \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\gamma ddz}{dx^2} &= \qquad\qquad\qquad + \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\delta d^3z}{dx^3} &= \qquad\qquad\qquad\qquad + \delta S\frac{d^4z}{dx^4} + \text{etc.}
\end{aligned}
$$

etc.

Since, having added all these values, they have to produce Sz, the coefficients α, β, γ, δ etc. will be defined from the following equations

$$
\begin{gathered}
\alpha - \frac{1}{2} = 0, \quad \beta - \frac{\alpha}{2} + \frac{1}{6} = 0, \quad \gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0,\\
\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0, \quad \varepsilon - \frac{\delta}{2} + \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0,\\
\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0 \quad \text{etc.}
\end{gathered}
$$

and continuing this way, one will find terms each second of which vanishes. Therefore, the third, fifth, seventh letter and in general all odd ones will be $= 0$ except for the first, which seems to violate the law of continuity. Therefore, it is even more necessary to prove rigorously that all odd terms except for the first necessarily vanish.

§113 Since each letter is determined according to a constant law from the preceding ones, they will constitute a recurring series. To make this explicit, assume this series

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.},$$

whose value we want to put $= V$, and it is obvious that this recurring series results from the expansion of this fraction

$$V = \frac{1}{1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{240}u^4 - \text{etc.}}$$

And if this fraction can be resolved into a power series in u in another way, it is necessary that always the same series

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.}$$

results; and this way another law, by which the same values α, β, γ, δ etc. are determined, will be found.

§114 Since, if e denotes the number, whose hyperbolic logarithm is equal to 1, it will be

$$e^{-u} = 1 - u + \frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{24}u^4 - \frac{1}{120}u^5 + \text{etc.},$$

it will also be

$$\frac{1 - e^{-u}}{u} = 1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \text{etc.}$$

and hence

$$V = \frac{u}{1 - e^{-u}}.$$

Now cancel the second term $\alpha u = \frac{1}{2}u$ from the series that

$$V - \frac{1}{2}u = 1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.};$$

it will be

$$V - \frac{1}{2}u = \frac{\frac{1}{2}u(1 + e^{-u})}{1 - e^{-u}}.$$

Multiply the numerator and denominator by $e^{\frac{1}{2}u}$ and it will be

$$V - \frac{1}{2}u = \frac{u\left(e^{\frac{1}{2}u} + e^{-\frac{1}{2}u}\right)}{\left(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u}\right)}$$

and, having converted the quantities $e^{\frac{1}{2}u}$ and $e^{-\frac{1}{2}u}$ into series, it will be

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\text{etc.}}{2\left(\frac{1}{2}+\frac{u^2}{2\cdot 4\cdot 6}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\text{etc.}\right)}$$

or

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^8}{2\cdot 4\cdots 12}+\frac{u^8}{2\cdot 4\cdots 16}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}+\frac{u^8}{4\cdot 6\cdots 14}+\frac{u^8}{4\cdot 6\cdots 18}+\text{etc.}}$$

§115 Therefore, because in this fraction the odd powers are completely missing, its power series will contain no odd powers at all; therefore, because $V-\frac{1}{2}u$ becomes equal to this series

$$1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

the coefficients of the odd powers, γ, ε, η, ι etc. will all vanish. And this is the reason why in the series $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\text{etc.}$ each second terms except for the first are $=0$ and the law of continuity is nevertheless not violated. Therefore, it will be

$$V=1+\frac{1}{2}u+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\varkappa x^{10}+\text{etc.}$$

and, having determined the letters β, δ, ζ, θ, $\varkappa$ etc. by expansion of the fraction above, we will obtain the summatory term Sz of the series whose general term corresponding to the index x is $=z$ expressed this way

$$Sz=\int zdx+\frac{1}{2}z+\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}+\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}+\text{etc.}$$

§116 Since the series $1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$ results from the expansion of this fraction

$$\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}+\frac{u^8}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}+\text{etc.}},$$

the letters β, δ, ζ, θ etc. will obey the following law

$$\beta = \frac{1}{2\cdot 4} - \frac{1}{4\cdot 6}$$

$$\gamma = \frac{1}{2\cdot 4\cdot 6\cdot 8} - \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10}$$

$$\delta = \frac{1}{2\cdot 4\cdot 6\cdots 12} - \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} - \frac{1}{4\cdot 6\cdots 11}$$

$$\theta = \frac{1}{2\cdot 4\cdot 6\cdots 16} - \frac{\zeta}{4\cdot 6} - \frac{\delta}{4\cdot 6\cdot 8\cdot 10} - \frac{\beta}{4\cdot 6\cdots 14} - \frac{1}{4\cdot 6\cdots 18}$$

etc.

But these values are positive and negative alternately.

§117 Therefore, if each second of these letters is assumed to be negative so that

$$Sz = \int z dx + \frac{1}{2}z - \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} - \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} - \text{etc.},$$

the letters β, δ, ζ, θ can be defined from this fraction

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^8}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} - \text{etc.}},$$

by expanding it into the series

$$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \text{etc.};$$

therefore, it will be

$$\beta = \frac{1}{4\cdot 6} - \frac{1}{2\cdot 4}$$

$$\delta = \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{2\cdot 4\cdot 6\cdot 8}$$

$$\zeta = \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{4\cdot 6\cdots 14} - \frac{1}{2\cdot 4\cdots 12}$$

$+$ etc.;

but now all terms will become negative.

§118 Therefore, let us put $\beta = -A$, $\delta = -B$, $\zeta = -C$ etc. such that

$$Sz = \int z dx + \frac{1}{2}z + \frac{Adz}{dx} - \frac{Bd^3z}{dx^3} + \frac{Cd^5z}{dx^5} - \frac{Dd^7z}{dx^7} + \text{etc.},$$

and to define the letters A, B, C, D etc., consider this series

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \text{etc.},$$

which results from the expansion of this fraction

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \frac{u^8}{2\cdot 4\cdots 16} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^6}{4\cdot 6\cdots 10} + \frac{u^8}{4\cdot 6\cdots 18} - \text{etc.}},$$

or consider this series

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.} = s,$$

which results from the expansion of this fraction

$$s = \frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.}}{u - \frac{u^3}{4\cdot 6} + \frac{u^5}{4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{4\cdot 6\cdots 14} + \text{etc.}}.$$

But since

$$\cos\frac{1}{2}u = 1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.},$$

$$\sin\frac{1}{2}u = \frac{u}{2} - \frac{u^3}{2\cdot 4\cdot 6} + \frac{u^5}{2\cdot 4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{2\cdot 4\cdots 14} + \text{etc.},$$

it follows that

$$s = \frac{\cos\frac{1}{2}u}{2\sin\frac{1}{2}u} = \frac{1}{2}\cot\frac{1}{2}u.$$

Therefore, if the cotangent of the arc $\frac{1}{2}u$ is converted into a series whose terms are ascending powers of u, from it one will find the values of the letters A, B, C, D, E etc.

§119 Therefore, since $s = \frac{1}{2}\cot\frac{1}{2}u$, it will be $\frac{1}{2}u = \operatorname{arccot} 2s$ and, by differentiating this equation, it will be $\frac{1}{2}du = \frac{-2ds}{1+4ss}$ or $4ds + du + 4ssdu = 0$ or

$$\frac{4ds}{du} + 1 + 4ss = 0.$$

But since

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - \text{etc.},$$

it will be

$$\begin{aligned}
\frac{4ds}{du} &= -\frac{4}{uu} - 4A - 3\cdot 4Bu^2 - 5\cdot 4Cu^4 - 7\cdot 4Du^6 - \text{etc.}\\
1 &= \phantom{-\frac{4}{uu}} \quad 1\\
4ss &= \frac{4}{uu} - 8A - 8Bu^2 - 8Cu^4 - 8Du^6 - \text{etc.}\\
&\qquad\qquad + 4A^2u^2 + 8ABu^4 + 8ACu^6 + \text{etc.}\\
&\qquad\qquad\qquad\qquad + 4BBu^6 + \text{etc.}
\end{aligned}$$

Having set these homogeneous terms equal to zero, it will be

$$A = \frac{1}{12}, \quad B = \frac{A^2}{5}, \quad C = \frac{2AB}{7}, \quad D = \frac{2AC + BB}{9}, \quad E = \frac{2AD + 2BD}{11},$$

$$F = \frac{2AE + 2BD + CC}{13}, \quad G = \frac{2AF + 2BE + 2CD}{15}, \quad H = \frac{2AG + 2BF + 2CE + DD}{17},$$

etc.

From these formulas it now obviously follows that each value is positive.

§120 But since the denominators of these values become immensely large and impede the calculation quite a lot, instead of the letters A, B, C, D etc. let us introduce these new letters

$$A=\frac{\alpha}{1\cdot 2\cdot 3},\quad B=\frac{\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5},\quad C=\frac{\gamma}{1\cdot 2\cdot 3\cdots 7},$$
$$D=\frac{\delta}{1\cdot 2\cdot 3\cdots 9},\quad E=\frac{\varepsilon}{1\cdot 2\cdot 3\cdots 11}\quad \text{etc.}$$

And one will find

$$\alpha=\frac{1}{2},\quad \beta=\frac{2}{3}\alpha^2,\quad \gamma=2\cdot\frac{2}{3}\alpha\beta,\quad \delta=2\cdot\frac{4}{3}\alpha\gamma+\frac{8\cdot 7}{4\cdot 5}\beta^2,$$
$$\varepsilon=2\cdot\frac{5}{3}\alpha\delta+2\cdot\frac{10\cdot 9\cdot 8}{1\cdot 2\cdots 5}\beta\gamma,\quad \zeta=2\cdot\frac{12}{1\cdot 2\cdot 3}\alpha\varepsilon+2\frac{12\cdot 11\cdot 10}{1\cdot 2\cdots 5}\beta\delta+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdots 7}\gamma\gamma,$$
$$\eta=2\cdot\frac{14}{1\cdot 2\cdot 3}\alpha\zeta+2\cdot\frac{14\cdot 13\cdot 12}{1\cdot 2\cdots 5}\beta\varepsilon+2\cdot\frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdots 7}\gamma\delta$$
$$\text{etc.}$$

§121 But we will apply these formulas more conveniently

$$\alpha=\frac{1}{2},\quad \beta=\frac{4}{3}\cdot\frac{\alpha\alpha}{2},\quad \gamma=\frac{6}{3}\cdot\alpha\beta,\quad \delta=\frac{8}{3}\cdot\alpha\gamma+\frac{8\cdot 7\cdot 6}{3\cdot 4\cdot 5}\cdot\frac{\beta\beta}{2},$$
$$\varepsilon=\frac{10}{3}\cdot\alpha\delta+\frac{10\cdot 9\cdot 8}{3\cdot 4\cdot 5}\cdot\beta\gamma,\quad \zeta=\frac{12}{3}\cdot\alpha\varepsilon+\frac{12\cdot 11\cdot 10}{3\cdot 4\cdot 5}\cdot\beta\delta+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\frac{\gamma\gamma}{2},$$
$$\eta=\frac{14}{3}\cdot\alpha\zeta+\frac{14\cdot 13\cdot 12}{3\cdot 4\cdot 5}\cdot\beta\varepsilon+\frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\gamma\delta,$$
$$\theta=\frac{16}{3}\cdot\alpha\eta+\frac{16\cdot 15\cdot 14}{3\cdot 4\cdot 5}\cdot\beta\zeta+\frac{16\cdot 15\cdots 12}{3\cdot 4\cdots 7}\gamma\varepsilon+\frac{16\cdot 15\cdots 10}{3\cdot 4\cdots 9}\cdot\frac{\delta\delta}{2}$$
$$\text{etc.}$$

Using this law, which simplifies the calculation a lot, if the values of the letters α, β, γ, δ etc. were found, then the summatory term of any arbitrary series whose general term or the term corresponding to the index x was $=z$, will be expressed as follows

$$Sz=\int zdx+\frac{1}{2}z+\frac{\alpha dz}{1\cdot 2\cdot 3\cdot dx}-\frac{\beta d^3z}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5}+\frac{\gamma d^5z}{1\cdot 2\cdots 7dx^5}$$

$$-\frac{\delta d^7 z}{1\cdot 2\cdots 9dx^7}+\frac{\varepsilon d^9 z}{1\cdot 2\cdots 11dx^9}-\frac{\zeta d^{11} z}{1\cdot 2\cdots 13dx^{11}}+\text{etc.}$$

But these letters α, β, γ, δ etc. were found to have the following values:

$$\begin{aligned}
\alpha &= \frac{1}{2} \quad \text{or} & 1\cdot 2\alpha &= 1\\
\beta &= \frac{1}{6} & 1\cdot 2\cdot 3\beta &= 1\\
\gamma &= \frac{1}{6} & 1\cdot 2\cdot 3\cdot 4\gamma &= 4\\
\delta &= \frac{3}{10} & 1\cdot 2\cdot 3\cdots 5\delta &= 36\\
\varepsilon &= \frac{5}{6} & 1\cdot 2\cdot 3\cdots 6\varepsilon &= 600\\
\zeta &= \frac{691}{210} & 1\cdot 2\cdot 3\cdots 7\zeta &= 24\cdot 691\\
\eta &= \frac{35}{2} & 1\cdot 2\cdot 3\cdots 8\eta &= 20160\cdot 35\\
\theta &= \frac{3617}{30} & 1\cdot 2\cdot 3\cdots 9\theta &= 12096\cdot 3617\\
\iota &= \frac{43867}{42} & 1\cdot 2\cdot 3\cdots 10\iota &= 86400\cdot 43867\\
\varkappa &= \frac{1222277}{110} & 1\cdot 2\cdot 3\cdots 11\varkappa &= 362880\cdot 1222277\\
\lambda &= \frac{854513}{6} & 1\cdot 2\cdot 3\cdots 12\lambda &= 79833600\cdot 854513\\
\mu &= \frac{1181820455}{546} & 1\cdot 2\cdot 3\cdots 13\mu &= 11404800\cdot 1181820455\\
\nu &= \frac{76977927}{2} & 1\cdot 2\cdot 3\cdots 14\nu &= 43589145600\cdot 76977927\\
\xi &= \frac{23749461029}{30} & 1\cdot 2\cdot 3\cdots 15\xi &= 43589145600\cdot 23749461029\\
\pi &= \frac{8615841276005}{462} & 1\cdot 2\cdot 3\cdots 16\pi &= 45287424000\cdot 8615841276005
\end{aligned}$$

§122 These numbers have the greatest use throughout the whole doctrine of series. For, first using these numbers one can form the last terms in the sums of the even powers, on which we remarked above [§ 63 of the first part] that they cannot be found in the same way as the remaining terms from the sums of the preceding power. For, in the even powers the last terms containing x of the sums are multiplied by certain numbers which numbers for the powers II, IV, VI, VII etc. are $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$ etc. affected with alternating signs. But these numbers result, if the values of the letters α, β, γ, δ etc. found above are respectively divided by the odd numbers 3, 5, 7, 9 etc. whence these numbers, which are usually called Bernoulli numbers after its discoverer Jacob Bernoulli, will be

$$
\begin{aligned}
\frac{\alpha}{3} &= \frac{1}{6} = \mathfrak{A}, & \frac{\iota}{19} &= \frac{43867}{798} = \mathfrak{I} \\
\frac{\beta}{5} &= \frac{1}{30} = \mathfrak{B}, & \frac{\varkappa}{21} &= \frac{174611}{330} = \mathfrak{K} = \frac{283 \cdot 617}{330} \\
\frac{\gamma}{7} &= \frac{1}{42} = \mathfrak{C}, & \frac{\lambda}{21} &= \frac{854513}{138} = \mathfrak{L} = \frac{11 \cdot 131 \cdot 593}{2 \cdot 3 \cdot 23} \\
\frac{\alpha}{9} &= \frac{1}{30} = \mathfrak{D}, & \frac{\mu}{25} &= \frac{236364091}{2730} = \mathfrak{M} \\
\frac{\varepsilon}{11} &= \frac{5}{66} = \mathfrak{E}, & \frac{\nu}{27} &= \frac{8553103}{6} = \mathfrak{N} = \frac{13 \cdot 657931}{6} \\
\frac{\zeta}{13} &= \frac{691}{2730} = \mathfrak{F}, & \frac{\xi}{27} &= \frac{23749461029}{870} = \mathfrak{O} \\
\frac{\eta}{15} &= \frac{7}{6} = \mathfrak{G}, & \frac{\pi}{31} &= \frac{8615841276005}{14322} = \mathfrak{P} \\
\frac{\theta}{17} &= \frac{3617}{510} = \mathfrak{H}, & & \text{etc.}
\end{aligned}
$$

§123 Therefore, one will be able to find these Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. immediately from the following equations

$$\begin{aligned}
\mathfrak{A} &= \frac{1}{6} \\
\mathfrak{B} &= \frac{4\cdot 3}{1\cdot 2}\cdot\frac{1}{5}\mathfrak{A}^2 \\
\mathfrak{C} &= \frac{6\cdot 5}{1\cdot 2}\cdot\frac{2}{7}\mathfrak{A}\mathfrak{B} \\
\mathfrak{D} &= \frac{8\cdot 7}{1\cdot 2}\cdot\frac{2}{9}\mathfrak{A}\mathfrak{C}+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{9}\mathfrak{B}^2 \\
\mathfrak{E} &= \frac{10\cdot 9}{1\cdot 2}\cdot\frac{2}{11}\mathfrak{A}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{13}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{1}{13}\mathfrak{C}^2 \\
\mathfrak{G} &= \frac{14\cdot 13}{1\cdot 2}\cdot\frac{2}{15}\mathfrak{A}\mathfrak{F}+\frac{14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{15}\mathfrak{B}\mathfrak{E}+\frac{14\cdot 13\cdot 12\cdot 12\cdot 11\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{2}{15}\mathfrak{C}\mathfrak{D} \\
&\text{etc.,}
\end{aligned}$$

the structure of which equations is clear per se, if one only notes, that where the square of a certain letter occurs, its coefficient is half as small as it seems to have to be according to the rule. But, the terms containing the products of different letters are to be considered to occur twice; for example it will be

$$\begin{aligned}
13\mathfrak{F} = \frac{12\cdot 11}{1\cdot 2}\mathfrak{A}\mathfrak{E}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\mathfrak{C}\mathfrak{C} \\
+\frac{12\cdot 11\cdot 10\cdots 5}{1\cdot 2\cdot 3\cdots 8}\mathfrak{D}\mathfrak{B}+\frac{12\cdot 11\cdot 10\cdots 3}{1\cdot 2\cdot 3\cdots 10}\mathfrak{E}\mathfrak{A}.
\end{aligned}$$

§124 Further, the same numbers α, β, γ, δ etc. enter the expressions of the sums of the series of fractions contained in this general form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.},$$

if n is a positive even number. For, we gave these sums expressed in terms of the half of the circumference of the circle π whose radius is $=1$ in the *Introductio* and these numbers α, β, γ, δ etc. are detected to enter the coefficients of these powers. But to understand that this does not happen accidentally but has to happen, let us investigate the same sums in a special way so that the structure of those sums will become clear more easily. Since we found above (§ 43) that

$$\frac{\pi}{n}\cot\frac{m}{n}\pi = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.},$$

combining each two terms, we will have

$$\frac{\pi}{n}\cot\frac{m}{n}\pi = \frac{1}{m} - \frac{2m}{nn-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \text{etc.},$$

whence we conclude

$$\frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \text{etc.} = \frac{1}{2mm} - \frac{\pi}{2mn}\cot\frac{m}{n}\pi.$$

Now, let us set $n = 1$ and for m let us put u that

$$\frac{1}{1-u^2} + \frac{1}{4-u^2} + \frac{1}{9-u^2} + \frac{1}{16-u^2} + \text{etc.} = \frac{1}{2uu} - \frac{\pi}{2u}\cot\pi u.$$

Resolve each of these fractions into series:

$$\begin{aligned}
\frac{1}{1-u^2} &= 1 + u^2 + u^4 + u^6 + u^8 + \text{etc.} \\
\frac{1}{4-u^2} &= \frac{1}{2^2} + \frac{u^2}{2^4} + \frac{u^4}{2^6} + \frac{u^6}{2^8} + \frac{u^8}{2^8} + \text{etc.} \\
\frac{1}{9-u^2} &= \frac{1}{3^2} + \frac{u^2}{3^4} + \frac{u^4}{3^6} + \frac{u^6}{3^8} + \frac{u^8}{3^{10}} + \text{etc.} \\
\frac{1}{16-u^2} &= \frac{1}{4^2} + \frac{u^2}{4^4} + \frac{u^6}{4^8} + \frac{u^8}{4^{10}} + \text{etc.} \\
&\text{etc.}
\end{aligned}$$

§125 Therefore, if one puts

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=\mathfrak{a} \qquad 1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=\mathfrak{b}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\mathfrak{c} \qquad 1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=\mathfrak{d}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathfrak{c} \qquad 1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=\mathfrak{f}$$

etc.

the above series will be transformed into this one

$$\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\mathfrak{e}u^8+\mathfrak{f}u^{10}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

Therefore, because in § 118 the letters A, B, C, D etc. were found to be of such a nature that, having put

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.},$$

$s=\frac{1}{2}\cot\frac{1}{2}u$, having written πu instead of $\frac{1}{2}u$ or $2\pi u$ instead of u, it will be

$$\frac{1}{2}\cot\pi u=\frac{1}{2\pi u}-2A\pi u-2^3B\pi^3u^3-2^5C\pi^5u^5-2^7D\pi^7u^7-\text{etc.},$$

whence, by multiplying by $\frac{\pi}{u}$, it will be

$$\frac{\pi}{2u}\cot\pi u=\frac{1}{2uu}-2A\pi^2-2^3B\pi^4u^2-2^5C\pi^6u^4-2^7D\pi^8u^6-\text{etc.},$$

and hence it follows that

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u=2A\pi^2+2^3B\pi^4u^2+2^5C\pi^6u^4+2^7D\pi^8u^6+\text{etc.}$$

Since we just found that

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u=\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\text{etc.},$$

it is necessary that

$$
\begin{aligned}
\mathfrak{a} &= 2 \quad A\pi^2 = \frac{2\alpha}{1\cdot 2\cdot 3}\pi^2 = \frac{2\mathfrak{A}}{1\cdot 2}\pi^2\\
\mathfrak{b} &= 2^3 \quad A\pi^4 = \frac{2^3\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^4 = \frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4\\
\mathfrak{c} &= 2^5 \quad A\pi^6 = \frac{2^5\gamma}{1\cdot 2\cdot 3\cdots 7}\pi^6 = \frac{2^3\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}\pi^6\\
\mathfrak{d} &= 2^7 \quad A\pi^8 = \frac{2^7\delta}{1\cdot 2\cdot 3\cdots 9}\pi^8 = \frac{2^5\mathfrak{D}}{1\cdot 2\cdot 3\cdots 8}\pi^8\\
\mathfrak{e} &= 2^9 \quad E\pi^{10} = \frac{2^9\varepsilon}{1\cdot 2\cdot 3\cdots 11}\pi^{10} = \frac{2^9\mathfrak{E}}{1\cdot 2\cdot 3\cdots 10}\pi^{10}\\
\mathfrak{f} &= 2^{11} F\pi^{12} = \frac{2^{11}\zeta}{1\cdot 2\cdot 3\cdots 13}\pi^{12} = \frac{2^{11}\mathfrak{F}}{1\cdot 2\cdot 3\cdots 12}\pi^{12}\\
&\text{etc.}
\end{aligned}
$$

§126 Therefore, by this simple reasoning not only all series of reciprocal powers we exhibited in the preceding paragraph are conveniently summed, but at the same time it is also understood how these sums are formed from the known values of the letters α, β, γ, δ, ε etc. or even from the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. Therefore, since we defined fifteen of these numbers in § 122, from these one will be able to assign the sums of all even [reciprocal] powers up to the sum of this series:

$$1+\frac{1}{2^{30}}+\frac{1}{3^{30}}+\frac{1}{4^{30}}+\frac{1}{5^{30}}+\text{etc.};$$

for, the sum of this series will be

$$=\frac{2^{29}\pi}{1\cdot 2\cdot 3\cdots 31}\pi^{31}=\frac{2^{29}\mathfrak{P}}{1\cdot 2\cdots 30}\pi^{30}.$$

And if one wants to determine more of these letters, this is very easily done by continuing these numbers α, β, γ etc. or these $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc.

§127 Therefore, the origin of these numbers α, β, γ, δ etc. or those formed from them $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. is basically the expansion of the cotangent of a certain angle into an infinite series. For, if

$$\frac{1}{2}\cot\frac{1}{2}u = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.},$$

it will be

$$Au^2 + Bu^4 + Cu^6 + Du^8 + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u;$$

therefore, if the respective values of the letters are substituted for coefficients A, B, C, D etc., it will be found

$$\frac{\alpha u^2}{1\cdot 2\cdot 3} + \frac{\beta u^4}{1\cdot 2\cdots 5} + \frac{\gamma u^7}{1\cdot 2\cdots 7} + \frac{\delta u^8}{1\cdot 2\cdots 9} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u$$

and by using the Bernoulli numbers it will be

$$\frac{\mathfrak{A}u^2}{1\cdot 2} + \frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}u^6}{1\cdot 2\cdots 6} + \frac{\mathfrak{D}u^8}{1\cdot 2\cdots 8} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u,$$

from which series by differentiation innumerable others can be deduced and infinite series these numbers enter can be summed.

§128 Let us take the first equation which we want to multiply by u so that

$$\frac{\alpha u^3}{1\cdot 2\cdot 3} + \frac{\beta u^5}{1\cdot 2\cdots 5} + \frac{\gamma u^7}{1\cdot 2\cdots 7} + \frac{\delta u^9}{1\cdot 2\cdots 9} + \text{etc.} = u - \frac{uu}{2}\cot\frac{1}{2}u,$$

which differentiated and divided by du gives

$$\frac{\alpha u^2}{1\cdot 2} + \frac{\beta u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\gamma u^6}{1\cdot 2\cdots 6} + \frac{\delta u^8}{1\cdot 2\cdots 8} + \text{etc.} = 1 - u\cot\frac{1}{2}u + \frac{uu}{4(\sin\frac{1}{2}u)^2};$$

and if it is differentiated again, it will be

$$\frac{\alpha u}{1} + \frac{\beta u^3}{1\cdot 2\cdot 3} + \frac{\gamma u^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.} = -\cot\frac{1}{2}u + \frac{u}{(\sin\frac{1}{2}u)^2} - \frac{uu\cos\frac{1}{2}u}{4(\sin\frac{1}{2}u)^2}.$$

But if the other equation is differentiated, it will be

$$\frac{\mathfrak{A}u}{1} + \frac{\mathfrak{B}u^3}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}u^5}{1\cdot 2\cdots 5} + \frac{\mathfrak{D}u^7}{1\cdot 2\cdots 7} + \text{etc.} = -\frac{1}{2}\cot\frac{1}{2}u + \frac{u}{4(\sin\frac{1}{2}u)^2}.$$

From these, if one puts $u = \pi$, because of $\cot \frac{1}{2}\pi = 0$ and $\sin \frac{1}{2}\pi = 1$, these summations follow

$$\begin{aligned}
1 &= \frac{\alpha\pi^2}{1\cdot 2\cdot 3} + \frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 7} + \frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 9} + \text{etc.} \\
1 + \frac{\pi^2}{4} &= \frac{\alpha\pi^2}{1\cdot 2} + \frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 8} + \text{etc.} \\
\pi &= \frac{\alpha\pi}{1} + \frac{\beta\pi^3}{1\cdot 2\cdot 3} + \frac{\gamma\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\delta\pi^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}
\end{aligned}$$

or

$$1 = \alpha + \frac{\beta\pi^2}{1\cdot 2\cdot 3} + \frac{\gamma\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\delta\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.};$$

if the first is subtracted from this one, it will remain

$$\alpha = \frac{(\alpha-\beta)\pi^2}{1\cdot 2\cdot 3} + \frac{(\beta-\gamma)\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{(\alpha-\gamma)\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

But then it will be

$$\begin{aligned}
1 &= \frac{\mathfrak{A}\pi^2}{1\cdot 2} + \frac{\mathfrak{B}\pi^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}\pi^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\mathfrak{D}\pi^8}{1\cdot 2\cdot 3\cdots 8} \\
\frac{\pi}{4} &= \frac{\mathfrak{A}\pi}{1} + \frac{\mathfrak{B}\pi^3}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\mathfrak{D}\pi^7}{1\cdot 2\cdot 3\cdots 7}
\end{aligned}$$

or

$$\frac{1}{4} = \frac{\mathfrak{A}}{1} + \frac{\mathfrak{B}\pi^2}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\mathfrak{D}\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

§129 From the table of values of the numbers α, β, γ, δ etc. we exhibited above (§ 121) it is plain that they decrease at first, but then increase and do so to infinity. Therefore, it will be worth one's while to investigate, how these numbers grow, after they had already been continued very far. Therefore, let φ be any number of this series of the numbers α, β, γ, δ etc. removed very far from the beginning and let ψ be the following number. Since the sum of the reciprocal powers are defined by means of these numbers, let $2n$ be the exponent of the power, whose sum the number φ enters; $2n+2$ will be the exponent corresponding to number ψ and the number n will already be immensely large. Hence, from § 125 one will have

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\text{etc.}=\frac{2^{2n-1}\varphi}{1\cdot 2\cdot 3\cdots(2n+1)}\pi^{2n},$$

$$1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\frac{1}{4^{2n+2}}+\text{etc.}=\frac{2^{2n+1}\varphi}{1\cdot 2\cdot 3\cdots(2n+3)}\pi^{2n+2}.$$

Therefore, if these numbers are divided by each other, it will be

$$\frac{1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\text{etc.}}{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{etc.}}=\frac{4\psi\pi^2}{(2n+2)(2n+3)\varphi}.$$

but since n is an immensely large number and since both series are very close to 1, it will be

$$\frac{\psi}{\varphi}=\frac{(2n+2)(2n+3)}{4\pi^2}=\frac{nn}{\pi\pi}.$$

Therefore, because n denotes, how far away the number φ was from the first number α, that number φ will have almost the same ratio to the following ψ as π^2 has to to n^2 which ratio will be exact, if n was an infinite number. Since it is almost $\pi\pi=10$, if one puts $n=100$, the hundredth term will be thousand times smaller than its following term. Therefore, the numbers α, β, γ, δ etc. as the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constitute a highly divergent series which grows even faster than a geometric series whose terms increase.

§130 Therefore, having found the values of the numbers α, β, γ, δ etc. or $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., if a series is propounded whose general term z was an arbitrary function of the index x, the summatory term Sz of this series will be expressed as follows

$$Sz=\int zdx+\frac{1}{2}z+\frac{1}{6}\cdot\frac{dz}{1\cdot 2}-\frac{1}{30}\cdot\frac{d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}$$
$$+\frac{1}{42}\cdot\frac{d^5}{1\cdot 2\cdot 3\cdots 6dx^5}-\frac{1}{30}\cdot\frac{d^7z}{1\cdot 2\cdot 3\cdots 8dx^7}$$

$$+\frac{5}{66}\cdot\frac{d^9z}{1\cdot2\cdot3\cdot10dx^9}-\frac{691}{2730}\cdot\frac{d^{11}z}{1\cdot2\cdot3\cdots12dx^{11}}$$
$$+\frac{7}{6}\cdot\frac{d^{13}z}{1\cdot2\cdot3\cdots14dx^{13}}-\frac{3617}{510}\cdot\frac{d^{15}z}{1\cdot2\cdot3\cdots16dx^{16}}$$
$$\frac{854513}{138}\cdot\frac{d^{21}z}{1\cdot2\cdot3\cdots22dx^{21}}-\frac{236364091}{2730}\cdot\frac{d^{23}z}{1\cdot2\cdot3\cdots24dx^{23}}$$
$$+\frac{8553103}{6}\cdot\frac{d^{25}z}{1\cdot2\cdot3\cdots26dx^{25}}-\frac{23749461029}{870}\cdot\frac{d^{27}z}{1\cdot2\cdot3\cdots28dx^{27}}$$
$$+\frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1\cdot2\cdot3\cdots30dx^{29}}-\text{etc.}$$

Therefore, if the integral $\int zdx$ or the quantity whose differential is $=zdx$ is known, the summatory term will be found by means of iterated differentiation. But it is to be noted that always a constant of such a kind is to be added to this expression that the sum becomes $=0$, if the index x is put $=0$.

§131 Therefore, if z was a polynomial function of x, since its higher order differentials vanish eventually, the summatory terms will be expressed by a finite expression; we will illustrate this in the following examples.

EXAMPLE 1

Let the summatory term of this series be in question

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & x \\ 1 + & 9 + & 25 + & 49 + & 81 & + & \cdots + & (2x-1)^2. \end{array}$$

Since here it is $z=(2x-1)^2=4xx-4x+1$, it will be

$$\int zdx=\frac{4}{3}x^3-2x^2+x;$$

for, from the differentiation of this series the equation $4xxdx-4xdx+dx=zdx$ results. Furthermore, by differentiation it will be

$$\frac{dz}{dx}=8x-4,\quad \frac{ddz}{dx^2}=8,\quad \frac{d^3z}{dx^3}=0\quad\text{etc.}$$

Hence the summatory term in question will be

$$\frac{4}{3}x^3 - 2x^2 + x + 2xx - 2x + \frac{1}{2} + \frac{2}{3}x \pm \text{Const.},$$

which constant has to cancel the terms $\frac{1}{2} - \frac{1}{3}$; therefore, it will be

$$S(2x-1)^2 = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{x}{3}(2x-1)(2x+1).$$

So, for $x = 4$ the sum of the first four terms will be

$$1 + 9 + 25 + 49 = \frac{4}{3} \cdot 7 \cdot 9 = 84.$$

EXAMPLE 2

Let the summatory term of this series be in question

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & & & x \\ 1 & + & 27 & + & 125 & + & 343 & + & \cdots & + & (2x-1)^3. \end{array}$$

Since $z = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$, it will be

$$\frac{dz}{dx} = 24x^2 - 24x + 6, \quad \frac{ddz}{dx^2} = 48x - 24, \quad \frac{d^3z}{dx^3} = 48;$$

the following differential quotients all vanish. Therefore, it will be

$$\begin{aligned} S(2x-1)^3 = 2x^4 - 4x^3 + 3x^3 - 1x & \\ + 4x^3 - 6x^2 + 3x - & \frac{1}{2} \\ + 2x^2 - 2x + & \frac{1}{2} \\ - & \frac{1}{15} \pm \text{Const.}, \end{aligned}$$

i.e.

$$S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1).$$

So, having put $x = 4$, it will be

$$1 + 27 + 125 + 343 = 16 \cdot 31 = 496.$$

§132 From this general expression found for the summatory term immediately that summatory term we gave in the above part [§ 29 and 61] for the powers of the natural numbers and whose demonstration could not be given at that point follows. For, if we put $z = x^n$, it will be $\int zdx = \frac{1}{n+1}x^{n+1}$; the differentials on the other hand will be

$$\frac{dz}{dx} = nx^{n-1}, \quad \frac{ddz}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3},$$

$$\frac{d^5z}{dx^5} = n(n-1)(n-2)(n-3)(n-4)x^{n-5}, \quad \frac{d^7z}{dx^7} = n(n-1)\cdots(n-6)x^{n-7} \quad \text{etc.}$$

From these therefore the following summatory term corresponding to the general term x^n will be deduced

$$
\begin{aligned}
Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1} - \frac{1}{30}\cdot\frac{n(n-1)(n-2)}{2\cdot3\cdot4}x^{n-3} \\
+\frac{1}{42}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot3\cdot4\cdot5\cdot6}\,x^{n-5} \\
-\frac{1}{30}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-6)}{2\cdot3\cdots8}x^{n-7} \\
+\frac{5}{66}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-8)}{2\cdot3\cdots10}x^{n-9} \\
-\frac{691}{2730}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-10)}{2\cdot3\cdots12}x^{n-11} \\
+\frac{7}{6}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-12)}{2\cdot3\cdots14}x^{n-13} \\
-\frac{3617}{510}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-14)}{2\cdot3\cdots16}x^{n-13} \\
+\frac{43867}{798}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-16)}{2\cdot3\cdots18}x^{n-17} \\
-\frac{174611}{330}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-18)}{2\cdot3\cdots20}x^{n-19} \\
+\frac{854513}{138}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-20)}{2\cdot3\cdots22}x^{n-21} \\
-\frac{236364091}{2730}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-22)}{2\cdot3\cdots24}x^{n-23} \\
+\frac{8553103}{6}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-24)}{2\cdot3\cdots26}x^{n-25} \\
-\frac{23749461029}{870}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-26)}{2\cdot3\cdots28}x^{n-27} \\
+\frac{8615841276005}{14322}\cdot\frac{n(n-1)\cdots\cdots\cdots\cdots(n-28)}{2\cdot3\cdots30}x^{n-29} \\
\text{etc.;}
\end{aligned}
$$

this expression does not differ from the one we gave above except for the fact that here we introduced the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., whereas above we used the numbers α, β, γ, δ etc.; nevertheless, the agreement is immediately clear. Therefore, it is possible to exhibit the summatory term of all series up to the sums of the thirtieth powers; this investigation, if it would been done in another way, would have required very long and most tedious calculations.

§133 Above (§ 59) we already gave an almost identical expression to define the summatory term from the general term. But that expressions used the iterated differences of the general term; therefore, it differs from the expression we gave here mainly in that regard that it does not require the integral $\int zdx$, but each difference of the general term is multiplied by certain functions of x. Therefore, let us find the same expression again in the following way more accommodated to the nature of series, from which at the same time the rule how the coefficients of the differentials proceed will be seen. Therefore, let the general term of the series be z, a function of the index x; the summatory term in question the other hand shall be $= s$; since this term, as we saw above, will be a function of x vanishing for $x = 0$, applying the results we demonstrated above [§ 68] on the nature of functions of this kind it will be

$$s - \frac{xds}{1dx} + \frac{x^2dds}{1\cdot 2dx^2} - \frac{x^3d^3s}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4s}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = 0.$$

§134 Since s denotes the sum of all terms of the series from the first to the last z, it is perspicuous, if in s one writes $x-1$ instead of x, that the first sum does not contain the last term z; it will be

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

and hence

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.},$$

which equation provides a way to define the general term from the given summatory term what is per se very easy. But from an appropriate combination of this equation with that we found in the preceding paragraph, one will be able to define the value of s in terms of x and z. For this aim, let us put

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Ddz^3}{dx^3} - \frac{Ed^4z}{dx^4} + \text{etc.} = 0,$$

where A, B, C, D etc. denote the necessary coefficients, either constant or variable; for, since

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \text{etc.},$$

if the values for z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. are substituted in the above equation, it will result

$$\begin{array}{rrllllll}
+ & s = s & & & & & & \\
- & Az = & -\frac{Ads}{dx} & + \frac{Adds}{2dx^2} & - \frac{Ad^3s}{6dx^3} & + \frac{Ad^4s}{24dx^4} & - \frac{Ad^5s}{120dx^5} & + \text{etc.} \\
+ & \frac{Bdz}{dx} = & & + \frac{Bdds}{dx^2} & - \frac{Bd^3s}{2dx^3} & + \frac{Bd^4s}{6dx^4} & - \frac{Bd^5s}{24dx^5} & + \text{etc.} \\
- & \frac{Cddz}{dx^2} = & & & - \frac{Cd^3s}{dx^3} & + \frac{Cd^4s}{2dx^4} & - \frac{Cd^5s}{6dx^5} & + \text{etc.} \\
+ & \frac{Dd^3z}{dx^3} = & & & & + \frac{Dd^4s}{dx^4} & - \frac{Dd^5s}{2dx^5} & + \text{etc.} \\
- & \frac{Ed^4z}{dx^4} = & & & & & - \frac{Ed^5s}{dx^5} & + \text{etc.}
\end{array}$$

etc.

which series all added up therefore will be equal zero.

§135 Therefore, since we found before that

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \text{etc.},$$

if the above equation is put equal to this one, the following defining equations of the letters A, B, C, D etc. will result

$$A = x, \quad B = \frac{x^2}{2} - \frac{A}{2}, \quad C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6},$$
$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}, \quad E = \frac{x}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120} \quad \text{etc..}$$

Therefore, having found the values of the letters A, B, C, D etc., from the general term z the summatory term $s = Sz$ will be determined in such a way that

$$Sz = Az - \frac{Bdz}{dx} + \frac{Cddz}{dx^2} - \frac{Dd^3z}{dx^3} + \frac{Ed^4z}{dx^4} - \frac{Fd^5z}{dx^5} + \text{etc.}$$

§136 But since

$$A = x, \quad B = \frac{1}{2}x^2 - \frac{1}{2}x, \quad C = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x,$$
$$D = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}xx \quad \text{etc.,}$$

it is clear that these coefficients are the same as those we had above (§ 59); hence that expression of the summatory term is the same as the one we found there and therefore it will be

$$A = Sx^0 = S1, \quad B = \frac{1}{1}Sx^1 - \frac{1}{1}x, \quad C = \frac{1}{2}Sx^2 - \frac{1}{2}x^2,$$
$$D = \frac{1}{6}Sx^3 - \frac{1}{6}x^3, \quad E = \frac{1}{24}Sx^4 - \frac{1}{24}x^4 \quad \text{etc.}$$

Therefore, it will be

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$
$$+\frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.}$$

But if $x = 0$ in the general term z, the term corresponding to the index $= 0$ will result; if it is put $= a$, it will be

$$a = z - \frac{xdz}{dx} + \frac{x^2ddz}{2dx^2} - \frac{x^3d^3z}{6dx^3} + \text{etc.}$$

and hence

$$\frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.} = z - a,$$

having substituted which value one will have

$$Sz = (x+1)z - a - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3} + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$

Therefore, having found the sums of the powers from a certain given general term, one can exhibit the summatory term corresponding to it.

§137 Therefore, since we found two expressions of the summatory term Sz for the general term z and one of the formulas contains the integral $\int zdx$, if these two expressions are equated, one will obtain the value of $\int zdx$ expressed by means of a series. For, since

$$\begin{aligned}\int zdx &= +\frac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}\\ &= (x+1)z - a - \frac{dz}{1dx}Sx + \frac{ddz}{1\cdot 2dx^2}Sx^2 - \frac{d^3z}{1\cdot 2\cdot 3}Sx^3 + \text{etc.,}\end{aligned}$$

it will be

$$\int zdx = \left(x+\frac{1}{2}\right)z - a - \frac{dz}{dx}\left(Sx + \frac{1}{2}\mathfrak{A}\right) + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}\left(Sx^3 - \frac{1}{4}\mathfrak{B}\right)$$

$$+\frac{d^4z}{24dx^4}Sx^4 - \frac{d^5z}{120dx^5}\left(Sx^5 + \frac{1}{6}\mathfrak{C}\right) + \frac{d^6z}{720dx^6}Sx^6 - \frac{d^7z}{5040dx^7}\left(Sx^7 - \frac{1}{8}\mathfrak{D}\right) + \text{etc.,}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. denote the Bernoulli numbers exhibited above (§ 122).

For the sake of an example, let $z = xx$; it will be $a = 0$, $\frac{dz}{dx} = 2x$ and $\frac{ddz}{2dx^2} = 1$; it will hence be

$$\int xxdx = \left(x+\frac{1}{2}\right)xx - 2x\left(\frac{1}{2}xx + \frac{1}{2}x + \frac{1}{12}\right) + 1\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right)$$

or $\int xxdx = \frac{1}{3}x^3$; but $\frac{1}{3}x^3$, having differentiated it, gives $xxdx$, of course.

§138 Therefore, there is a new way to find the summatory terms of the series of powers; for, since from the coefficients A, B, C, D etc. assumed before these summatory terms are formed very easily, but each of these coefficients is conflated of the preceding ones, if the values given in § 136 are substituted for these letters in the formulas given in § 135, it will be

$$Sx^1 - x = \frac{1}{2}xx - \frac{1}{2}x$$

$$Sx^2 - x^2 = \frac{1}{3}x^3 - \frac{1}{3} \quad - \frac{2}{2}(Sx - x)$$

$$Sx^3 - x^3 = \frac{1}{4}x^4 - \frac{1}{4}x - \frac{3}{2}(Sx^2 - x^2) - \frac{3 \cdot 2}{2 \cdot 3}(Sx - x)$$

$$Sx^4 - x^4 = \frac{1}{5}x^5 - \frac{1}{5}x - \frac{4}{2}(Sx^3 - x^3) - \frac{4 \cdot 3}{2 \cdot 3}(Sx^2 - x^2) - \frac{4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}(Sx - x)$$

etc.

Therefore, one will be able to form the sums of the above powers from the sums of the lower ones.

§139 But if we consider the law, which the coefficients A, B, C, D etc. above (§ 135) were found to follow, with more attention, we will detect that they constitute a recurring series. For, if we expand this fraction

$$y = \frac{x + \frac{1}{2}xxu + \frac{1}{6}x^3u^2 + \frac{1}{24}x^4u^3 + \frac{1}{120}x^5u^4 + \text{etc.}}{1 + \frac{1}{2}u \quad + \frac{1}{6}u^2 \quad + \frac{1}{24}u^3 \quad + \frac{1}{120}u^4 \quad + \text{etc.}}$$

into a power series in u and assume this series to result

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \text{etc.},$$

it will be, as we found before,

$$A = x, \quad B = \frac{1}{2}xx - \frac{1}{2}A \quad \text{etc.}$$

and so having found this series one will obtain the summatory terms of the series of powers. But that fraction, from whose expansion this series results, will go over into this form $\frac{e^{xu}-1}{e^u-1}$ which, if x was a positive integer, goes over into

$$1 + e^u + e^{2u} + e^{3u} + \cdots + e^{(x-1)u};$$

therefore, since

$$\begin{aligned}
1 &= 1 \\
e^u &= 1 + \frac{u}{1} + \frac{u^2}{1\cdot 2} + \frac{u^3}{1\cdot 2\cdot 3} + \frac{u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} \\
e^{2u} &= 1 + \frac{2u}{1} + \frac{4u^2}{1\cdot 2} + \frac{8u^3}{1\cdot 2\cdot 3} + \frac{16u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} \\
e^{3u} &= 1 + \frac{3u}{1} + \frac{9u^2}{1\cdot 2} + \frac{27u^3}{1\cdot 2\cdot 3} + \frac{81u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} \\
e^{(x-1)u} &= 1 + \frac{(x-1)u}{1} + \frac{(x-1)^2u^2}{1\cdot 2} + \frac{(x-1)^3u^3}{1\cdot 2\cdot 3} + \frac{(x-1)^4u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}
\end{aligned}$$

it will be

$$\begin{aligned}
A &= x \\
B &= S(x-1) = Sx - x \\
C &= \frac{1}{2}S(x-1)^2 = \frac{1}{2}Sx^2 - \frac{1}{2}x^2 \\
D &= \frac{1}{6}S(x-1)^3 = \frac{1}{6}Sx^3 - \frac{1}{6}x^3 \\
&\text{etc.}
\end{aligned}$$

Therefore, the connection already mentioned before of these coefficients to the sums of the powers is confirmed and demonstrated completely.